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A GENERALIZED COUETTE-TYPE FLOW WITH VARIABLE VISCOSITY

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Аннотация—Рассматривается течение капельной жидкости с переменной вязкостью в плоском и кольцевом каналах, через поверхности которых впрыскивается или отсасывается жидкость. Получено решение динамической и тепловой задач для случая, когда течение обусловлено наличием продольного градиента давления и движением одной из поверхностей канала.

NOMENCLATURE

a, thermal diffusivity;
A, constant;
C_f, resistance coefficient;
p, pressure;
P, dimensionless pressure;
P_0, dimensionless pressure which causes the flow to separate;
Pe, Péclet number;
Q, dimensionless blowing parameter;
r, radial co-ordinate;
R_1, R_2, internal and external cylinder radii, respectively;
T, temperature;
u, v, longitudinal and transverse velocity components;
U, surface velocity;
x, y, cartesian co-ordinates;
$Y(\eta)$, $= QRe\int_0^\eta \frac{d\eta}{\nu_*}$;
Y_1, $= QRe\int_0^1 \frac{d\eta}{\nu_*}$;
h, distance between plates;
Re, Reynolds number.

Greek symbols

β, $= \frac{R_2}{R_1}$;
η, dimensionless transverse co-ordinate;
ν, kinematic viscosity;
ν_1, kinematic viscosity at the fixed surface;
ν_*, dimensionless kinematic viscosity;
π, dimensionless radial pressure gradient;
ρ, fluid density.

1. INTRODUCTION

FLUID flows between two porous surfaces have been the subject of a number of works [1, 2, etc.] in which all physical properties of the fluid were assumed constant.

In the present paper the solution of the Navier–Stokes and energy equations is presented for two cases of liquid flow: between two permeable plane parallel surfaces, and in an annulus between two coaxial cylinders. The fluid viscosity is assumed to depend on temperature. As to the other liquid properties (density, thermal conductivity, etc.), they are assumed constant because of their weak dependence on temperature.

2. PLANE FLOW

We shall consider a liquid flow in a plane channel of height h, formed by two permeable surfaces. The flow is initiated by the motion of one surface with the constant velocity U and longitudinal pressure gradient. It is assumed that the liquid is injected (or sucked) through the lower surface at a rate constant along the surface. The velocity components and temperature are assumed to depend on the co-ordinate y only. Then it follows from the continuity equation that v = const. throughout, and the problem is reduced to integration of the set of equations

$$v\frac{du}{dy} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{d}{dy}\left(\nu\frac{du}{dy}\right) \tag{1}$$

$$\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$v\frac{dT}{dy} = a\frac{d^2T}{dy^2} \tag{3}$$

with boundary conditions

$$u = 0, \quad T = T_1 \quad \text{at} \quad y = 0$$
$$u = U, \quad T = T_2 \quad \text{at} \quad y = h \tag{4}$$

The solution of the set of equations (1), (2)

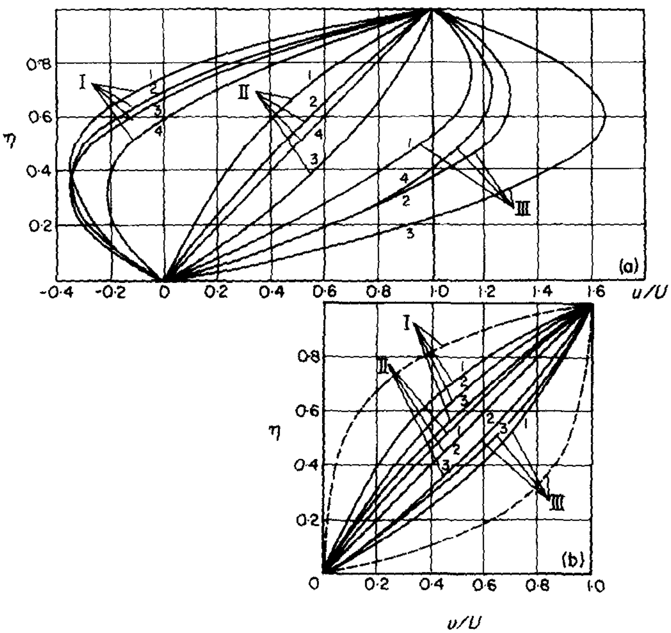

FIG. 1.

(a) Velocity distribution in a plane channel with different values of pressure gradient.

I. $P = -5$; II. $P = 0$; III. $P = +5$.
1. $QRe = +1{\cdot}4$; 2. $QRe = 0$; 3. $QRe = -1{\cdot}4$; 4. $QRe = 0$; $T_1 = T_2 = 50°C$.

(Curves 1, 2, 3 correspond to temperatures of plates $T_1 = 50°C$, $T_2 = 80°C$).

(b) Velocity distributions (solid lines) and temperature (dashed lines) in plane channel with porous walls without longitudinal pressure gradient. $T_1 = 50°C$ in all cases.

I. $QRe = +1{\cdot}4$; II. $QRe = 0$; III. $QRe = -1{\cdot}4$.
1. $T_2 = 80°C$; 2. $T_2 = 50°C$; 3. $T_2 = 20°C$.

and (3) may be written in the following dimensionless form

$$\frac{u}{U} = \frac{\exp y(\eta) - 1}{\exp y(1) - 1}\left\{1 + \frac{P}{Q}\left[\exp y(1) \int_0^{\eta} \exp(-y)\,d\eta - 1\right]\right\} - \frac{P}{Q}\left[\exp y(\eta) \int_0^{\eta} \exp(-y)\,d\eta - \eta\right], \quad (5)$$

$$\frac{T - T_1}{T_2 - T_1} = \frac{1 - \exp(QPe\eta)}{1 - \exp(QPe)} \quad (6)$$

Here the following notations are introduced:

$$P = \frac{1}{\rho U^2}\left(-\frac{dp}{dx}\right), \quad y(\eta) = \left[\int_0^{\eta} \frac{d\eta}{\nu_*}\right] QRe,$$

$$Re = \frac{Uh}{\nu_1}, \quad Pe = \frac{Uh}{a}, \quad Q = \frac{v}{U},$$

$$\eta = \frac{y}{h}, \quad \nu_* = \frac{\nu}{\nu_1}, \quad \nu_1 = \nu(T_1).$$

The curves in Fig. 1 show velocity and temperature distributions predicted by formulae (5) and (6) for water flowing in a plane channel. The relation between viscosity and temperature was taken from the experimental data of reference [3].

Comparison of velocity distributions corresponding to isothermal and non-isothermal flows allows us to understand the effect of viscosity change on the formation of a velocity field. In a non-isothermal flow, separation at one of the plates may occur in an adverse longitudinal pressure gradient as in the case of an isothermal flow. The condition of flow separation on the stationary plate which follows from the equation $du/dy|_{y=0} = 0$ is of the form

$$P_0 = Q\left[\exp y_1 \int_0^1 \exp(-y)\,d\eta - 1\right]^{-1} \quad (7)$$

(P_0 is the dimensionless pressure gradient which causes the flow to separate).

It can be seen from Fig. 2(a), that separation at the lower plate may occur at both injection and suction of liquid. As to the effect of non-uniformity of the temperature field, conclusion may be made that the positive temperature gradient $dT/dy|_{y=0} > 0$ provokes separation.

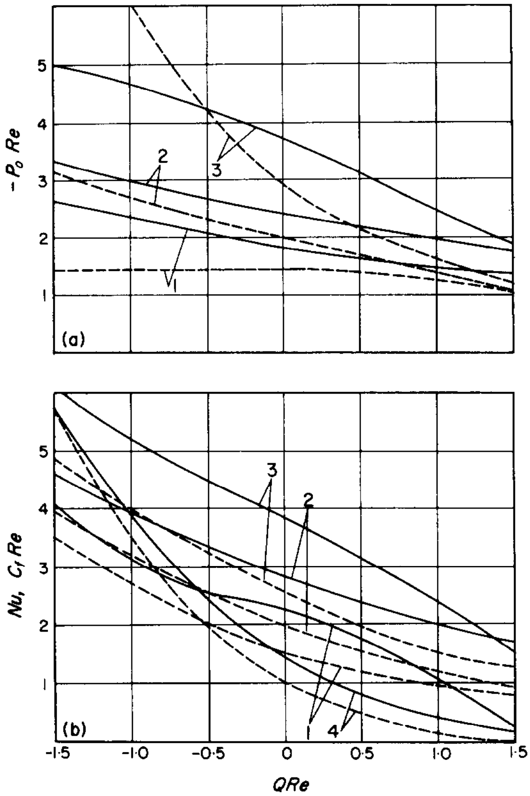

FIG. 2.

(a) Longitudinal pressure gradient P_0 which causes the flow to separate.

——— flow between coaxial cylinders ($\beta = 0{\cdot}5$),
– – – flow in a plane channel

1. $T_2 = 80°C$
2. $T_2 = 50°C$
3. $T_2 = 20°C$

$T_1 = 50°C$.

(b) Heat-transfer and resistance coefficients in flow between permeable walls.

——— in an annulus ($\beta = 0{\cdot}5$);
– – – between plane walls.

1. $T_2 = 80°C$
2. $T_2 = 50°C$
3. $T_2 = 20°C$
4. Nu

at $T_1 = 50°C$; $C_f Re$; $P = 0$.

If, as it is done above, the dissipation function in the energy equation is neglected, then the Nusselt number is independent of the viscosity. Contrary to this, the resistance coefficient

$$(C_f \, . \, Re)_{y=0} = \frac{2QRe}{\exp y_1 - 1}$$

depends essentially on the viscosity change [Fig. 2(b)]: when the temperature gradient is positive, resistance at the wall will be smaller than that in an isothermal flow.

3. AXISYMMETRICAL FLOW

We turn to the flow in an annulus between two coaxial porous cylinders with radii R_1 and R_2. We shall assume that the internal cylinder is stationary and the external cylinder is moving with the velocity U. Let the velocity components and temperature be functions of the co-ordinate r only. Then from the continuity equation

$$\frac{dv}{dr} + \frac{v}{r} = 0$$

it follows that

$$v = \frac{A}{r} \qquad \text{where} \quad A = \text{const.} \tag{8}$$

Thus the flow is governed by the Navier–Stokes and energy equations of the form

$$\frac{A}{r}\frac{du}{dr} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{r}\frac{d}{dr}\left(\nu r \frac{du}{dr}\right), \tag{9}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{A^2}{r^3} - 2\frac{A}{r^2}\frac{dv}{dr}, \tag{10}$$

$$\frac{A}{r}\frac{dT}{dr} = \frac{a}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) \tag{11}$$

with boundary conditions

$$u = 0, \quad T = T_1 \quad \text{at} \quad r = R_1$$
$$u = U, \quad T = T_2 \quad \text{at} \quad r = R_2. \tag{12}$$

Since $\partial p/\partial r$ is independent of x, then

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial p}{\partial x}\right) = 0.$$

Consequently, $\partial p/\partial x$ is independent of r, but it follows then from equation (9) that $\partial p/\partial x =$ const.

Making use of the dimensionless quantities

$$\eta = \frac{r}{R_1}; \qquad \beta = \frac{R_2}{R_1}; \qquad P = \frac{1}{\rho U^2}\left(-\frac{\partial p}{\partial x}\right);$$
$$\pi = \frac{1}{\rho U^2}\frac{\partial p}{\partial r}; \qquad Q = \frac{A}{\nu_1}; \qquad \nu_* = \frac{\nu}{\nu_1};$$
$$Re = \frac{UR_1}{\nu_1}; \qquad Pe = \frac{UR_1}{a};$$
$$y(\eta) = \left[\int_1^\eta \frac{d\eta}{\nu_* \eta}\right] . \, QRe; \qquad \nu_1 = \nu(T_1)$$

the solutions of equations (9) to (11) may be written in the following form

$$\frac{u}{U} = \frac{\exp y(\eta) - 1}{\exp y(\beta) - 1}\left\{1 + \frac{P}{Q}\left[\exp y(\beta) \int_1^\beta \eta \exp(-y)\, d\eta - \frac{\beta^2 - \exp y(\beta)}{2}\right]\right\}$$
$$- \frac{P}{Q}\left[\exp y(\eta) \int_1^\eta \eta \exp(-y)\, d\eta - \frac{\eta^2 - \exp y(\eta)}{2}\right]. \tag{13}$$

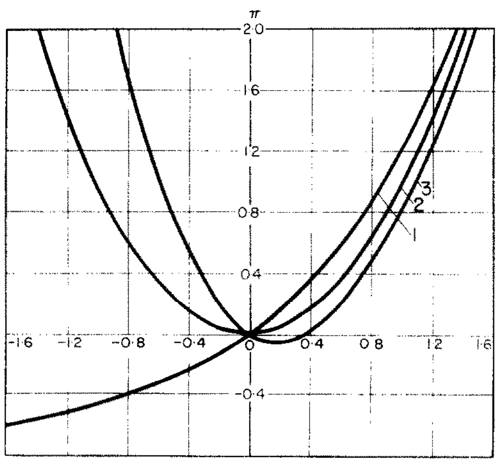

FIG. 3. Radial pressure gradient in an annulus ($\beta = 0{\cdot}5$) with porous walls ($\partial p/\partial x = 0$).

1. $T_2 = 80°C$
2. $T_2 = 50°C$
3. $T_2 = 20°C$

$T_1 = 50°C$.

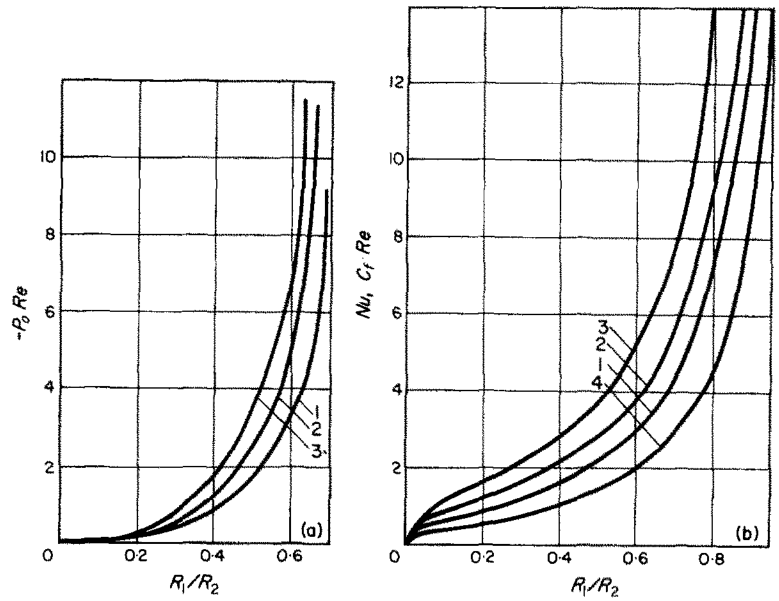

FIG. 4.

(a) Longitudinal pressure gradient in an annulus which causes the flow to separate ($Q = 0$).

1. $T_2 = 80°C$
2. $T_2 = 50°C$ $\Big\}$ $T_1 = 50°C$.
3. $T_2 = 20°C$

(b) Heat-transfer and resistance coefficients in an annulus ($\beta = 0{\cdot}5$).

1. $T_2 = 80°C$
2. $T_2 = 50°C$
3. $T_2 = 20°C$ $\Big\}$ $T_1 = 50°C$; $C_f Re$.
4. Nu

$$\frac{T - T_1}{T_2 - T_1} = \frac{1 - \eta^{QPe}}{1 - \beta^{QPe}} \tag{14}$$

$$\pi = \frac{Q^2}{\eta^3}\left(\frac{1}{\eta} - \frac{2}{QRe}\frac{\mathrm{d}v_*}{\mathrm{d}\eta}\right). \tag{15}$$

Velocity and temperature distributions as well as their deformation due to change of viscosity in the flow field are completely analogous quantitatively to those considered above for the plane problem.

The pressure gradient which causes the flow to separate is defined by the following expression

$$P_0 = 2Q[1 - \beta^2 + 2\exp y(\beta) \int_1^\beta \eta \exp(-y)\,\mathrm{d}\eta]^{-1} \tag{16}$$

It is not difficult to calculate the resistance and heat-transfer coefficients using equations (13) and (14).

The curves in Fig. 3 allow assessment of the effect of the temperature non-uniformity of the flow on the transverse pressure gradient on the internal cylinder. It should be noted that in the case of an isothermal flow ($v_* = 1$), $\partial p/\partial r|_{r=R_1} > 0$ both for injection and suction. Contrary to this, in a non-isothermal flow the value and sign of the pressure gradient $\partial p/\partial r|_{r=R_1}$ depend not only on the value of the radial velocity at the internal cylinder, but also on its direction [see equation (15)].

The dependence of heat transfer and resistance coefficients as well as longitudinal pressure gradient at the external cylinder which causes the flow to separate, on the injection (or suction) rate is the same as in a plane flow. The effect of non-uniformity of the temperature field is analogous in both cases.

Parameters of the problem of a flow with axial symmetry include a new quantity: the ratio of cylinder radii. This is a specific feature of the flow of such a type.

Figure 4 shows that heat transfer, resistance and longitudinal pressure gradient at the internal cylinder increase as the ratio $\beta = R_1/R_2$ grows, and when the ratio β is the same, these quantities increase with the temperature ratio T_1/T_r.

REFERENCES

1. G. M. Lilley, On a generalized porous-wall "Couette-type" flow, *J. Aero/Space Sci.* **26**, 685–686 (1959).
2. N. T. Dunwoody, On shearing flow between porous coaxial cylinder, *J. Aerospace Sci.* **29**, 494–495 (1962).
3. S. S. Kutateladze and V. M. Borishansku, *Handbook of Heat Transmission* (Spravochnik po teploperedache). Gosenergoizdat, Moscow–Leningrad (1959).

Abstract—In the present paper, a study is made of liquid flow with variable viscosity in plane and annular channels, liquid being injected or sucked through their porous surfaces. The solution is obtained of the dynamic and thermal problems for the case in which the flow is caused by a longitudinal pressure gradient and motion of one of the channel surfaces.

Résumé—On étudie l'écoulement d'un liquide à viscosité variable dans des conduites de section rectangulaires et annulaires, le liquide étant injecté ou aspiré à travers leurs parois poreuses. La solution des problèmes dynamique et thermique est obtenue dans le cas où l'écoulement est produit par un gradient longitudinal de pression et le mouvement d'une des parois de la conduite.

Zusammenfassung—In der vorliegenden Arbeit wird eine Untersuchung der Flüssigkeitsströmung mit veränderlicher Zähigkeit in ebenen und ringförmigen Kanälen durchgeführt, wobei Flüssigkeit durch die porösen Wände zu- oder abgeführt wird. Die Lösung für die dynamischen und thermischen Probleme gilt für den Fall, dass die Strömung von einem Druckgradienten in Längsrichtung verursacht wird und sich eine Kanalwand bewegt.